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HOPF GALOIS EXTENSIONS AND AZUMAYA ALGEBRAS

LIANYONG XUE

Department of Mathematics
Bradley University
Peoria, Illinois 61625
USA
e-mail: lxue@bradley.edu

Abstract

Let H be a finite dimensional Hopf algebra over a field k, H^* be the dual Hopf algebra of H, and A be a left H-module algebra such that A^H is a direct summand of A as an A^H-bimodule. An equivalent condition is given for an Azumaya algebra A being an H^*-Galois extension of A^H, and several characterizations of a general H^*-Galois Azumaya extension B with center C are given in terms of the Azumaya smash product $B \# H$, where B is called a general H^*-Galois Azumaya extension of B^H, if it is an H^*-Galois extension (not necessarily separable) of an Azumaya C^H-algebra B^H.

1. Introduction

Let H be a finite dimensional Hopf algebra over a field k, H^* be the dual Hopf algebra of H, A be a left H-module algebra with center C, and

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A^H be the H-invariants of A. Assume A is an H^*-Galois extension of A^H such that A^H is a direct summand of A as a left A^H-module. In [9], Ulbrich gives an equivalent condition for A being an Azumaya algebra such that $C \subset A^H$; that is, A is an Azumaya algebra such that $C \subset A^H$, if and only if $A \supset V_A(A^H)$ and $A \supset A^H$ are Hirata separable extensions ([9], Theorem 2.7). In the present paper, we first strengthen the above result of Ulbrich when A^H is a direct summand of A as an A^H-bimodule by an expression of the Azumaya algebra $A \cong A^H \otimes_{C^H} V_A(A^H)$ over C^H, and then give an equivalent condition for an Azumaya algebra A such that $C \subset A^H$ being an H^*-Galois extension of A^H. Next, let B be a left H-module algebra with center C. Then, we study an Azumaya smash product $B \# H$ over C^H. We shall show that $B \# H$ is an Azumaya C^H-algebra, if and only if B is an H^*-Galois extension (not necessarily separable) of an Azumaya C^H-algebra B^H. Such a B is called a general H^*-Galois Azumaya extension, a generalization of an H^*-Galois Azumaya extension ([6]) and a Galois Azumaya extension for rings ([1]). Several characterizations of a general H^*-Galois Azumaya extension are given in terms of the smash product $B \# H$.

Throughout of this paper, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε, H^* the dual Hopf algebra of H, B a left H-module algebra, C the center of B, $B^H = \{b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H\}$ called the H-invariants of B, and $B \# H$ the smash product of B and H, where $B \# H = B \otimes_k H$ such that for all $b \# h$ and $b' \# h'$ in $B \# H$, $(b \# h)(b' \# h') = \sum b(h_1 b') \# h_2 h'$, where $\Delta(h) = \sum h_1 \otimes h_2$. As given in [5], B is called a (right) H^*-Galois extension of B^H, if B is a (right) H^*-comodule algebra with structure map $\rho : B \longrightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \longrightarrow B \otimes_k H^*$ is a bijection, where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

Following the definitions as given in [10] and [11], let B be a ring with 1 and a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. We call B a separable extension of A, if there exist $\{a_i, b_i$ in B, $i = 1, 2, \ldots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum ba_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B, where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A, if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. An H^*-Galois extension B is called an H^*-Galois Azumaya extension, if B is a separable H^*-Galois extension of B^H such that B^H is an Azumaya algebra over C^H ([6]), and a general H^*-Galois Azumaya extension, if B is a H^*-Galois extension (not necessarily separable) of B^H such that B^H is an Azumaya algebra over C^H. Throughout, an H^*-Galois extension means a right H^*-Galois extension unless it is stated otherwise.

2. Hopf Galois Extensions and Azumaya Algebras

Let A be a left H-module algebra with center C. We first strengthen the Ulbrich result as given by Theorem 2.7 in [9] when A^H is an A^H-bimodule direct summand of A and with center C^H.

Theorem 2.1. *Let A be an H^*-Galois extension of A^H. Assume A^H is a direct summand of A as an A^H-bimodule with center C^H. Then $A \supset V_A(A^H)$ and $A \supset A^H$ are Hirata separable extensions, if and only if $A \cong A^H \otimes_{C^H} V_A(A^H)$ as Azumaya C^H-algebras (hence $C = C^H$).*

Proof. ($\Rightarrow$) Since A is a Hirata separable extension of A^H, which is a direct summand of A as an A^H-bimodule, $V_A(A^H)$ is a separable C-algebra ([8], Proposition 1.3) and $V_A(V_A(A^H)) = A^H$ ([7], Proposition 1.2). This implies that the center of A is contained in A^H and the center of A^H is the same as the center of $V_A(A^H)$. Hence $C = C^H =$ the center of A^H by hypothesis. Since A is also a Hirata separable extension of $V_A(A^H)$, A is a separable C^H-algebra by the transitivity property of separable extensions. Thus, both A^H and $V_A(A^H)$ are Azumaya C^H-algebra. Therefore,

$$A \cong V_A(A^H) \otimes_{C^H} V_A(V_A(A^H)) \text{ ([2], Theorem 4.3, page 57)}$$

$$= V_A(A^H) \otimes_{C^H} A^H \cong A^H \otimes_{C^H} V_A(A^H),$$

as Azumaya C^H-algebra.

($\Leftarrow$) Since $A \cong A^H \otimes_{C^H} V_A(A^H)$ as Azumaya C^H-algebras, $C = C^H$ and A is finitely generated and projective over A^H and $V_A(A^H)$, respectively. Thus A is a Hirata separable extension of A^H and $V_A(A^H)$, respectively ([3], Theorem 1).

We recall that A is called an H^*-Galois Azumaya extension of A^H, if A is a separable H^*-Galois extension over A^H, which is an Azumaya algebra over C^H ([6], Theorem 3.4).

Corollary 2.2. *Let A be an H^*-Galois extension as given by Theorem* 2.1. *Then A is an H^*-Galois Azumaya extension of A^H.*

Proof. By Theorem 2.1, $A \cong A^H \otimes_{C^H} V_A(A^H)$ as Azumaya C^H-algebras, so A is a separable H^*-Galois extension of the Azumaya C^H-algebra A^H.

Next, we give an equivalent condition for an Azumaya algebra A such that $C \subset A^H$ being an H^*-Galois extension of A^H.

Theorem 2.3. *Let A be an Azumaya C-algebra with* $C \subset A^H$, *and* A^H *is a direct summand of A as an* A^H*-bimodule with center* C^H. *Then, A is an* H^**-Galois extension of* A^H, *if and only if* $A \# H$ *is an Azumaya* C^H*-algebra. For such an* A, $A \# H \cong A^H \otimes_{C^H} (V_A(A^H) \# H)$ *as Azumaya* C^H*-algebras.*

Proof. ($\Rightarrow$) By Theorem 2.1, $A \cong A^H \otimes_{C^H} V_A(A^H)$ as Azumaya C^H-algebras, so A^H and $V_A(A^H)$ are Azumaya C^H-algebras ([2], Theorem 4.4, page 58). Also, by Corollary 2.2, $V_A(A^H)$ is an H^*-Galois extension of C^H ([4], Lemma 2.8). Hence $V_A(A^H)$ is a finitely generated and projective C^H-module. Noting that C^H is a commutative ring contained in $V_A(A^H)$, we have that $V_A(A^H)$ is a generator of C^H. This implies that $\mathrm{Hom}_{C^H}(V_A(A^H), V_A(A^H))$ is an Azumaya C^H-algebra ([2], Proposition 4.1, page 56). But then $A \# H \cong A^H \otimes_{C^H} (V_A(A^H) \# H) \cong A^H \otimes_{C^H} \mathrm{Hom}_{C^H}(V_A(A^H), V_A(A^H))$, which is an Azumaya C^H-algebra.

($\Leftarrow$) Since $A \# H$ is an Azumaya C^H-algebra and a finitely generated and projective left A-module, it is a Hirata separable extension of A ([3], Theorem 1). But A is a generator of itself, so it is a generator of $A \# H$ ([3], Lemma). Thus A is an H^*-Galois extension of A^H ([6], Theorem 2.2).

3. General H^*-Galois Azumaya Extensions

In this section, we study the class of general H^*-Galois Azumaya extensions B; that is, B is an H^*-Galois extension (not necessarily separable) of B^H such that B^H is an Azumaya C^H-algebra, where C is the center of B. Several characterizations of such a B are given in terms of the smash product $B \# H$.

Theorem 3.1. *Let B be a left H-module algebra. Then the following statements are equivalent*:

(1) *B is a general H^*-Galois Azumaya extension.*

(2) *$B \# H$ is an Azumaya C^H-algebra.*

(3) *$B \# H$ is a Hirata separable extension of B and $V_{B \# H}(B)$, respectively.*

(4) *$B \# H$ is a Hirata separable extension of $V_{B \# H}(B)$ and $V_{B \# H}(B)$ is a left H^*-Galois extension of C^H.*

Proof. (1) $\Rightarrow$ (2) Since B is an H^*-Galois extension of B^H, $B \# H \cong \mathrm{Hom}_{B^H}(B, B)$, where B is a finitely generated and projective right B^H-module ([6], Theorem 2.2). Also B^H is an Azumaya C^H-algebra, so it is a finitely generated and projective C^H-module. Hence B is a finitely generated and projective C^H-module. Noting that C^H is a commutative ring contained in B, we have that B is a generator of C^H, so $\mathrm{Hom}_{C^H}(B, B)$ is an Azumaya C^H-algebra ([2], Proposition 4.1, page 56). By hypothesis, B^H is an Azumaya subalgebra of $\mathrm{Hom}_{C^H}(B, B)$. But then $V_{\mathrm{Hom}_{C^H}(B, B)}(B^H) \cong \mathrm{Hom}_{B^H}(B, B) \cong B \# H$ is an Azumaya C^H-algebra.

(2) $\Rightarrow$ (1) Since $B \# H$ is an Azumaya C^H-algebra and a finitely generated and projective left B-module, $B \# H$ is a Hirata separable extension of B ([3], Theorem 1). Hence B is a left generator of $B \# H$ because B is a generator of itself ([3], Lemma). Thus B is an H^*-Galois extension of B^H ([6], Theorem 2.2). Moreover, since B is a direct summand of $B \# H$ as a left C^H-module, it is a finitely generated and projective C^H-module. Noting that C^H is a commutative ring contained in B, we have that B is a generator of C^H. This implies that $\mathrm{Hom}_{C^H}(B, B)$ is an Azumaya C^H-algebra containing an Azumaya subalgebra $B \# H$. Thus, the commutator of $B \# H$ in $\mathrm{Hom}_{C^H}(B, B)$ is an Azumaya C^H-algebra, that is, $B^H(\cong \mathrm{Hom}_{B \# H}(B, B))$ is an Azumaya C^H-algebra.

(2) $\Leftrightarrow$ (3) Since $B \# H$ is an H-Galois extension of B ([6], Example, page 45) such that B is a direct summand of $B \# H$ as a left B-module, $V_{B \# H}(V_{B \# H}(B)) = B$ ([7], Proposition 1.2). This implies that the center of $B \# H$ is C^H. But then $B \# H$ is an Azumaya C^H-algebra such that $C^H \subset B(= (B \# H)^{H^*})$, if and only if $B \# H$ is a Hirata separable extension of B and $V_{B \# H}(B)$, respectively ([9], Theorem 2.7).

(3) $\Rightarrow$ (4) Since $B\# H$ is an H-Galois extension of B again, we have that $B\# H$ is a Hirata separable extension of B, if and only if $V_{B\# H}(B)$ is a left H^*-Galois extension of the center of $B\# H$ ([9], Theorem 2.6). Now $B\# H$ is a Hirata separable extension of B and contains B as a direct summand as a left B-module, so $V_{B\# H}(V_{B\# H}(B)) = B$ ([7], Proposition 1.2). This implies that the center of $B\# H$ is C^H. Thus $V_{B\# H}(B)$ is a left H^*-Galois extension of C^H.

(4) $\Rightarrow$ (3) Since $B\# H$ is an H-Galois extension of B and $V_{B\# H}(B)$ is a left H^*-Galois extension of C^H, $V_{B\# H}(V_{B\# H}(B)) = B$ ([9], Lemma 2.5). This implies that the center of $B\# H$ is C^H. Thus $B\# H$ is a Hirata separable extension of B ([9], Theorem 2.6).

By Theorem 3.1, we derive a characterization of an H^*-Galois Azumaya extension (for more, see Theorem 3.4 in [6]).

Theorem 3.2. *Let B be a left H-module algebra. Then B is an H^*-Galois Azumaya extension of B^H, if and only if $B\# H$ is a separable extension of $V_{B\# H}(B)$, $V_{B\# H}(B)$ is a separable left H^*-Galois extension of C^H, and C^H is the center of $B\# H$.*

Proof. ($\Rightarrow$) Since $B\# H$ is an Azumaya C^H-algebra ([6], Theorem 3.4), it is a Hirata separable extension of B and $V_{B\# H}(B)$, respectively, by Theorem 3.1 (2) $\Rightarrow$ (3). Hence $B\# H$ is a separable extension of $V_{B\# H}(B)$ and $V_{B\# H}(B)$ is a left H^*-Galois extension of C^H by Theorem 3.1 (2) $\Rightarrow$ (4). Moreover, B is separable C^H-subalgebra of $B\# H$, so $V_{B\# H}(B)$ is a separable extension of C^H ([2], Theorem 4.3, page 57).

($\Leftarrow$) By the transitivity property of separable extensions, $B \# H$ is a separable C^H-algebra, so $B \# H$ is an Azumaya C^H-algebra. Moreover, noting that $V_{B \# H}(B)$ is a separable C^H-algebra by hypothesis, we have that $B(= V_{B \# H}(V_{B \# H}(B)))$ is also a separable C^H-algebra. Therefore, B is an H^*-Galois Azumaya extension of B^H ([6], Theorem 3.4).

Now, let B be a general H^*-Galois Azumaya extension of B^H, $\mathcal{S} = \{E \subset B^H | E$ is a separable C^H-algebra$\}$ and $\mathcal{T} = \{A \subset B | V_B(B^H) \subset A$, A is a general H^*-Galois Azumaya extension of A^H such that $A \# H$ is a separable C^H-algebra$\}$. We next show that $\alpha : \mathcal{S} \to \mathcal{T}$ by $\alpha(E) = V_B(E)$ for $E \in \mathcal{S}$ is a bijection. We need a lemma for separable subalgebras of an Azumaya algebra.

Lemma 3.3. *Let A be an Azumaya C-algebra and E be a separable subalgebra of A. Then the center of E and $V_A(E)$ are the same.*

Proof. This is a consequence of the commutator theorem for Azumaya algebras, $V_A(V_A(E)) = E$.

Theorem 3.4. *Let B be a general H^*-Galois Azumaya extension of B^H. Then, there exists a one-to-one correspondence between the set $\mathcal{S}$ and the set $\mathcal{T}$.*

Proof. Let $\alpha : E \longrightarrow V_B(E)$ for a separable C^H-subalgebra E of B^H. Then E is a separable subalgebra of the Azumaya C^H-algebra $B \# H$ by Theorem 3.1. Hence $V_{B \# H}(E)(= V_B(E) \# H)$ is also a separable C^H-subalgebra of $B \# H$ containing $V_B(B^H) \# H$. By Lemma 3.3, E and $V_B(E) \# H$ have the same center denoted by D. Thus $V_B(E) \# H$ is an Azumaya D-algebra. Moreover, since B^H is an Azumaya

C^H-algebra, $V_{B^H}(E)$ is an Azumaya D-subalgebra of B^H by Lemma 3.3 again. Now $V_B(E)$ is a left H-module algebra such that $(V_B(E))^H = V_{B^H}(E)$ (for $E \subset B^H$) and $V_B(E) \# H$ is an Azumaya D-algebra, so $V_B(E) (= \alpha(E))$ is a general H^*-Galois Azumaya extension containing $V_B(B^H)$ by Theorem 3.1. Thus $\alpha : \mathcal{S} \to \mathcal{T}$ is well defined.

Next, we claim that α is onto. Let A be a general H^*-Galois Azumaya extension in B containing $V_B(B^H)$ such that $A \# H$ is a separable C^H-algebra. Since $V_{B\#H}(B^H) = V_B(B^H)\# H$ and $V_{B\#H}(B^H) \subset A \# H$ such that $A \# H$ is a separable C^H-algebra, $V_{B\#H}(A \# H)$ is also a separable C^H-subalgebra of $B \# H$ and

$$V_{B\#H}(A \# H) \subset V_{B\#H}(V_B(B^H) \# H) = V_{B\#H}(V_{B\#H}(B^H)) = B^H$$

([2], Theorem 4.3, page 57). Hence $V_{B\#H}(A \# H)$ is a separable C^H-subalgebra of B^H ([2], Theorem 4.3, page 57). Denoting $V_{B\#H}(A \# H)$ by E, we have that $V_{B\#H}(E) = V_{B\#H}(V_{B\#H}(A \# H)) = A \# H$ ([2], Theorem 4.3, page 57). Since $E \subset B^H$, $V_{B\#H}(E) = V_B(E) \# H$. Thus $A \# H = V_B(E) \# H$; and so $A = V_B(E) = \alpha(E)$, where E is a separable C^H-subalgebra of B^H. This proves that α is onto. To show that α is one-to-one, assume that $E, F \in \mathcal{S}$ such that $\alpha(E) = \alpha(F)$, that is, $V_B(E) = V_B(F)$. Then $F = V_{B\#H}(V_{B\#H}(F)) = V_{B\#H}(V_B(F) \# H) = V_{B\#H}(V_B(E) \# H) = V_{B\#H}(V_{B\#H}(E)) = E$. Thus α is one-to-one. This completes the proof.

We conclude the paper with a general H^*-Galois Azumaya extension but not an H^*-Galois Azumaya extension as given in ([6], page 49 and [5], page 8). Let $H_4 = k\langle 1, g, x, gx \mid g^2 = 1, x^2 = 0, xg = -gx\rangle$, where k is a field of characteristic $\neq 2$, $B = H_4$, and $H = H_4^*$. Then B is an H^*-Galois extension of k but not separable over k. Hence B is a general H^*-Galois Azumaya extension but not an H^*-Galois Azumaya extension of k.

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